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IDENTIFICATION OF A CAVITY IN AN ELASTIC ROD IN THE ANALYSIS OF TRANSVERSE VIBRATIONS

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The direct and inverse problems of the steady-state transverse vibrations of a cylindrical rod with a defect in the form of a cavity of small relative size are considered. An approach to determining the location and volume of the cavity of arbitrary shape is proposed. Results of computational experiments are analyzed.

Key words: *identification, vibrations, cavity, regularization, rod.*

This paper considers the problem of steady-state transverse vibrations of a cylindrical rod containing a defect in the form of a cavity of small relative size. An analysis is made of the direct problem in which the presence of the cavity is modeled by relations between the geometrical parameters of the problem and the coordinates. The region occupied by the rod is broken into a finite number of subregions with constant characteristics. A numerical-analytical study is performed. Relations between the first resonant frequencies and the characteristic parameters of the cavity (coordinate of the center, volume) are obtained. These relations are analyzed, and the ranges of frequencies in which these relations are one-to-one are determined. We note that similar relations for the case of longitudinal vibrations of a cylindrical rod are investigated in [1], and an asymptotic analysis of the displacement of resonant frequencies for small cavities is performed in [2, 3]. The information obtained is used to formulate inverse problems of reconstruction of cavities of small relative size.

The present paper considers the inverse problem of the greatest practical importance consisting of determining the location and volume of a cavity irrespective of the shape of its surface. We note that an algorithm for the solution of similar inverse problems is presented in [4]. The proposed scheme has been used in a series of computational experiments aimed at reconstructing cavities of various shapes. The results of these experiments indicate a high effectiveness of the proposed approach.

1. DIRECT PROBLEM

1.1. Formulation of the Direct Problem. We consider steady-state transverse vibrations of a rod of length l with a cavity of arbitrary shape. We assume that one end of the rod ($x = 0$) is rigidly fixed and a time-periodic force $P(t) = P_0 \exp(-i\omega t)$ acts on the other end ($x = l$). The presence of the cavity is modeled by dependences of the region F and the moment of inertia J of the cross section on the longitudinal coordinate x. The cavity dimensions are considered small compared to the characteristic geometrical dimension of the cross section of the rod. It is required to determine the displacement of the nonfixed end of the rod (as a function of the vibration frequency) and the first resonant frequencies.

Assuming that the cavity dimensions are small, we use the equation of transverse vibrations of a rod of variable cross section [5]. In the case the steady-state vibrations, we obtain the following boundary-value problem for a differential operator of fourth order with variable coefficients:

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$$(J(x)w''(x))'' - k^2F(x)w(x) = 0,$$
$$w(0) = 0, \quad w'(0) = 0, \quad w''(l) = 0, \quad E(J(x)w''(x))'(l) = P_0. \tag{1.1}$$

Here $k^2 = \rho\omega^2/E$, ρ is the density, ω is the vibration frequency, and E is Young's modulus.

Problem (1.1) for arbitrary coefficients of the differential operator cannot be solved analytically; therefore, to construct its general solution, it is necessary to use numerical methods [6]. In [3], solution of a similar direct problem is reduced to solution of the Fredholm integral equation of the second kind.

1.2. Numerical Algorithm of Solving the Direct Problem. In the modeling of the localized defect, the cross-sectional area is assumed to vary as follows:

$$F(x) = F_0(1 - \eta(x)); \tag{1.2}$$

the function $\eta(x)$ is different from zero in the interval $[a, b] \subset [0, l]$. We divide the segment $[0, l]$ into n nonuniform parts:

$$x_0 = 0, \qquad x_i = a + \frac{b-a}{n-2}(i-1), \quad i = \overline{1, n-1}, \qquad x_n = l.$$

The grid is refined in the vicinity of the cavity. Let us assume that, in the ith interval $[x_{i-1}, x_i]$, the functions $F(x)$ and $J(x)$ have constant values: $F_i = F((x_{i-1} + x_i)/2)$ and $J_i = J((x_{i-1} + x_i)/2)$. In this case, the displacement function $w_i(x)$ in the ith interval satisfies the fourth-order equations with constant coefficients

$$w_i^{(\mathrm{IV})}(x) - \lambda_i^4 w_i(x) = 0, \qquad x \in [x_{i-1}, x_i], \quad i = \overline{1, n}, \tag{1.3}$$

where $\lambda_i^4 = k^2F_i/J_i$. The general solution of the ith equation in (1.3) is given by

$$w_i(x) = C_{1i}(\cosh(\lambda_i x) + \sinh(\lambda_i x)) + C_{2i}\sin(\lambda_i x) + C_{3i}\cos(\lambda_i x) + C_{4i}(\cosh(\lambda_i x) - \sinh(\lambda_i x)).$$

The constants C_{ji} are found from the boundary conditions and the conjugation conditions

$$w_1(0) = 0, \qquad w_1'(0) = 0,$$
$$w_i(x_i) = w_{i+1}(x_i), \qquad w_i'(x_i) = w_{i+1}'(x_i),$$
$$J_i w_i''(x_i) = J_{i+1}w_{i+1}''(x_i), \qquad J_i w_i'''(x_i) = J_{i+1}w_{i+1}'''(x_i),$$
$$w_n''(l) = 0, \qquad w_n'''(l) = p,$$

where $p = P_0/(EJ_0)$. It is obvious that the required function of the displacement of the free end of the rod is a function $w_n(l) = f(\omega)$ that depends on the vibration frequency. We note that refining the grid in the vicinity of the cavity allows a considerable increase in the accuracy of the solution of the direct problem. It is obvious that the accuracy of the solution increases with increasing n. The proposed algorithm was tested on the problem of a cylindrical rod of length $l = 1$ and radius $r_0 = l/5$ without a cavity. The displacement function for a homogeneous rod is known and is given by the expression

$$w_0(x) = -(p/\lambda^3)[U_1V_4(\lambda x) + U_2V_3(\lambda x)], \tag{1.4}$$

where $\lambda^4 = k^2F_0/J_0$, $U_1 = V_1(\lambda l)/U$, $U_2 = -V_2(\lambda l)/U$, $U = V_1^2(\lambda l) - V_2(\lambda l)V_4(\lambda l)$, and V_1, V_2, V_3, and V_4 are known Krylov functions [5]. In all intervals of the parameter λ not containing resonant frequencies, the relative error of the approximate solution does not exceed 1%; therefore, the proposed numerical algorithm can be used to solve similar problems.

To study the dependence of the first resonant frequencies on the parameters of the cavity, we introduce the dimensionless quantities $\gamma_i = \lambda_i^r l$, $\alpha = c/l$ and v (λ_i^r is the parameter λ calculated from the ith resonant frequency; c is the coordinate of the center of the cavity on the rod axis; v is the relative volume of the cavity). Figure 1 shows curves of $\gamma_i(\alpha)$ for the first three resonances. It is obvious that one-to-one relationship between the resonant frequency and the location of the cavity occurs only in the case of the first resonance.

Figure 2 shows curves of $\psi_i = \gamma_i(v)/\gamma_i^0$ for the first three resonances (γ_i^0 corresponds to the ith resonance of the homogeneous rod; $\alpha = 0.5$). It is obvious that all three resonances have one-to-one relationship between the resonant frequency and the volume of the cavity. Similar relationship hold for all values of α in the interval $(0, l)$.

It should be noted that inverse relations between the first resonant frequency and the location and volume of the cavity are single-valued functions and can serve as input information in determining the parameters of the cavity.

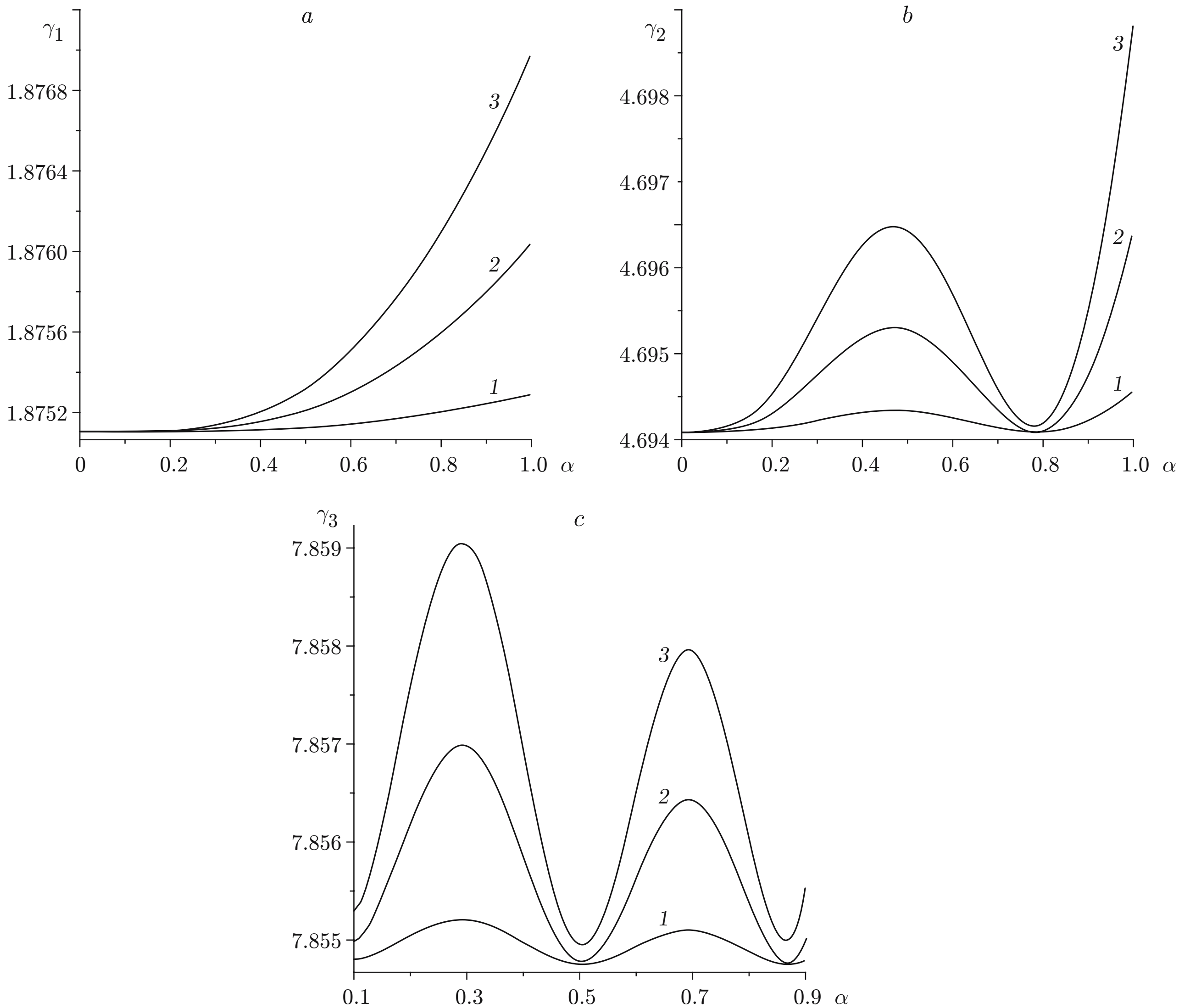

Fig. 1. Curves of the first (a), second (b), and third (c) resonances versus the location of the cavity $v = 0.0001$ (1), 0.0005 (2), and 0.001 (3).

2. INVERSE PROBLEM

2.1. Formulation of the Inverse Problem. In the inverse problem, the function of the displacement of the free end of the rod $f(\omega)$ is assumed to be defined in a certain range of the frequency $\omega \in [\omega_1, \omega_2]$. It is required to determine the location and volume of the cavity.

Let us consider the boundary-value problem (1.1). Assuming that the parameters of the cavity are small, we apply a linearization procedure. We set

$$w(x) = w_0(x) + \varepsilon w_1(x), \qquad F(x) = F_0(1 - \varepsilon\eta(x)), \qquad J(x) = J_0(1 - \varepsilon^2 q\eta^2(x)) \tag{2.1}$$

(ε is a formal small parameter). The function $\eta(x)$ modeling the presence of the cavity satisfies the conditions

$$\eta(x) \geqslant 0 \quad \forall\, x \in [0, l], \qquad \|\eta(x)\|_{L_2[0,l]} \ll 1.$$

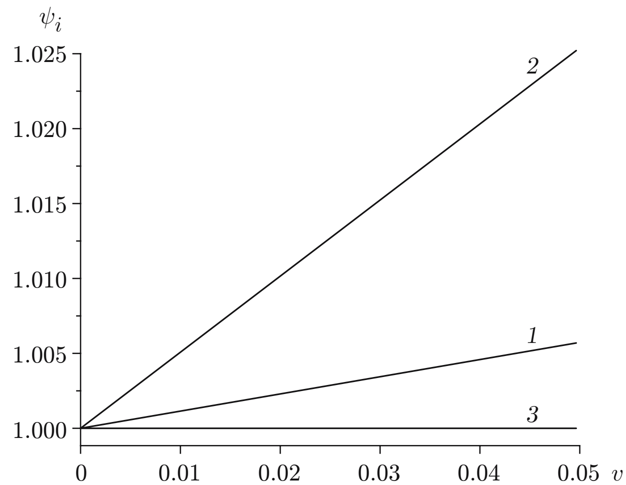

Fig. 2. First three resonances versus the volume of the cavity: 1) first resonance; 2) second resonance; 3) third resonance.

Substituting expressions (2.1) into the differential equation of problem (1.1) and equating the coefficients at the same powers of ε, according to [2], we obtain

$$\varepsilon^0: \quad w_0^{(\mathrm{IV})} - \lambda^4 w_0 = 0; \tag{2.2}$$

$$\varepsilon^1: \quad w_1^{(\mathrm{IV})} - \lambda^4 w_1 + \eta\lambda^4 w_0 = 0. \tag{2.3}$$

We multiply Eq. (2.2) by w_1 and Eq. (2.3) by w_0 and integrate over the segment $[0, l]$. Equating the integrals in the expressions obtained, we have the equation

$$\lambda^4 \int\limits_0^l \eta w_0^2 \, dx + \Big(\frac{\partial^3 w_1}{\partial x^3} \, w_0 - \frac{\partial^2 w_1}{\partial x^2} \, \frac{\partial w_0}{\partial x} + \frac{\partial w_1}{\partial x} \, \frac{\partial^2 w_0}{\partial x^2} - w_1 \, \frac{\partial^3 w_0}{\partial x^3} \Big) \Big|_0^l = 0.$$

Taking into account the boundary conditions

$$w_0(0) = 0, \qquad w_0'(0) = 0, \qquad w_0''(l) = 0, \qquad w_0'''(l) = p,$$

$$w_1(0) = 0, \qquad w_1'(0) = 0, \qquad w_1''(l) = 0, \qquad w_1'''(l) = 0$$

and the equality $f(\omega) = w_0(l) + w_1(l)$, we obtain the Fredholm integral equation of the first kind [7] for the function $\eta(x)$:

$$\lambda^4 \int\limits_0^l \eta(x) w_0^2(x) \, dx = p[f(\omega) - w_0(l)], \qquad \omega \in [\omega_1, \omega_2]. \tag{2.4}$$

We note that, because of the smoothness of the kernel of the integral equation, constructing a solution of the integral equation (2.4) is an ill-posed problem; therefore, regularization is required [8].

2.2. Regularization of the Inverse Problem. The problem of determining the location and volume of the cavity can be solved for the class of cylindrical cavities which is a compact in the space $L_2[0, l]$. Therefore, in solving the integral equation (2.4), we use the method of regularization on compact sets [9]. We set that the variation in the cross-sectional area of the rod as a function of the longitudinal coordinate x has the form of (1.2). In the case of a rod of circular cross section, $F_0 = \pi r_0^2$ and the function $\eta(x)$ is represented as

$$\eta(x) = (r/r_0)^2 H(h^2 - (x - c)^2), \tag{2.5}$$

where H is a Heaviside function, c is the coordinate of the center of the cavity, and r and $2h$ are the radius and height of the cylinder, respectively. We introduce the notation $r_1 = (r/r_0)^2$. In view of (2.5), Eq. (2.4) becomes

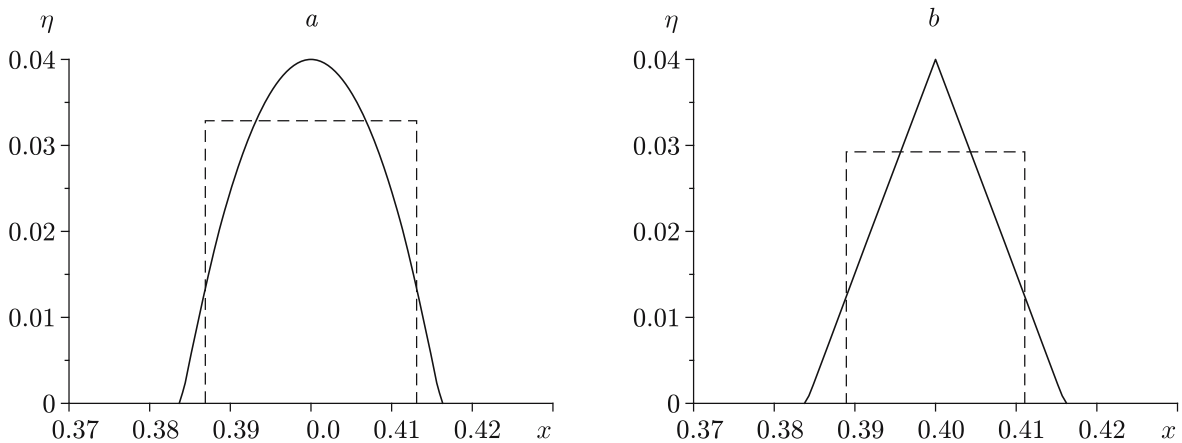

Fig. 3. Reconstruction of ellipsoidal cavity ($\varepsilon_v = 0.82$) (a) and conical cavity ($\varepsilon_v = 0.79$) (b) in the class of cylindrical cavities: the solid curve refers to the initial cavity and the dashed curve refers to the reconstructed cavity.

TABLE 1

Relative Errors ε_v and ε_α

Shape of cavity	$v = 0.0002$			$v = 0.0015$		
	$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 0.9$
Cylindrical	$\varepsilon_\alpha = 0.31$ $\varepsilon_v = 0.72$	$\varepsilon_\alpha = 0.03$ $\varepsilon_v = 0.07$	$\varepsilon_\alpha = 0.02$ $\varepsilon_v = 0.01$	$\varepsilon_\alpha = 1.21$ $\varepsilon_v = 2.82$	$\varepsilon_\alpha = 0.13$ $\varepsilon_v = 0.29$	$\varepsilon_\alpha = 0.09$ $\varepsilon_v = 0.06$
Ellipsoidal	$\varepsilon_\alpha = 0.27$ $\varepsilon_v = 0.72$	$\varepsilon_\alpha = 0.02$ $\varepsilon_v = 0.04$	$\varepsilon_\alpha = 0.04$ $\varepsilon_v = 0.05$	$\varepsilon_\alpha = 0.75$ $\varepsilon_v = 1.80$	$\varepsilon_\alpha = 0.08$ $\varepsilon_v = 0.17$	$\varepsilon_\alpha = 0.06$ $\varepsilon_v = 0.04$
Conical	$\varepsilon_\alpha = 0.14$ $\varepsilon_v = 0.33$	$\varepsilon_\alpha = 0.02$ $\varepsilon_v = 0.03$	$\varepsilon_\alpha = 0.01$ $\varepsilon_v = 0.01$	$\varepsilon_\alpha = 0.60$ $\varepsilon_v = 1.40$	$\varepsilon_\alpha = 0.07$ $\varepsilon_v = 0.14$	$\varepsilon_\alpha = 0.04$ $\varepsilon_v = 0.03$

$$\lambda^4 r_1^2 \int\limits_{c-h}^{c+h} w_0^2(x)\,dx = p[f(\omega) - w_0(l)], \qquad \omega \in [\omega_1, \omega_2]. \tag{2.6}$$

Substitution of the solution of the homogeneous problem (1.4) into expression (2.6) yields the following functional equation for three parameters of the cavity:

$$r_1^2[U_1^2 R_1(c,h,\lambda) + 2U_1U_2R_{12}(c,h,\lambda) + U_2^2 R_2(c,h,\lambda)] = (4\lambda^2/p)[f(\omega) - w_0(l)], \quad \omega \in [\omega_1, \omega_2]. \tag{2.7}$$

Here

$$\begin{gathered} R_{1,2}(c,h,\lambda) = h \pm h + (2\lambda)^{-1}[\cosh\,(2\lambda c)\sinh\,(2\lambda h) \pm \cos\,(2\lambda c)\sin\,(2\lambda h)] \\ - (2/\lambda)\{\cos\,(\lambda c)\cosh\,(\lambda c)[\sin\,(\lambda h)\cosh\,(\lambda h) \pm \cos\,(\lambda h)\sinh\,(\lambda h)] \\ + \sin\,(\lambda c)\sinh\,(\lambda c)[\cos\,(\lambda h)\sinh\,(\lambda h) \mp \sin\,(\lambda h)\cosh\,(\lambda h)]\}, \\ R_{12}(c,h,\lambda) = (2\lambda)^{-1}[\sinh\,(2\lambda c)\sinh\,(2\lambda h) + \sin\,(2\lambda c)\sin\,(2\lambda h)] \\ - (2/\lambda)[\cos\,(\lambda c)\sinh\,(\lambda c)\sin\,(\lambda h)\cosh\,(\lambda h) + \sin\,(\lambda c)\cosh\,(\lambda c)\cos\,(\lambda h)\sinh\,(\lambda h)]. \end{gathered} \tag{2.8}$$

Taking into account that the parameter h is small, we apply the average theorem to the integral in Eq. (2.6). As a result, we obtain

$$\int\limits_{c-h}^{c+h} w_0^2(x)\,dx = 2hw_0^2(c) + o(h^2). \tag{2.9}$$

In view of (2.9), Eq. (2.6) reduces to the relation

$$vw_0^2(c) = p[f(\omega) - w_0(l)]/(\lambda^4 l), \qquad \omega \in [\omega_1, \omega_2], \tag{2.10}$$

where $v = 2hr_1^2/l$ is the relative volume of the cavity. We note that a similar result is obtained by decomposition of expressions (2.8) in a Taylor series in the parameter h in the vicinity of zero.

Thus, the problem of reconstructing a cavity of small relative size reduces to determining two parameters of the cavity from the functional equation (2.10).

The proposed approach was used to calculate a series of computational experiments aimed at reconstructing cavities of various shapes. Solution of Eq. (2.7) allows reconstructing any cavities in the class of cylindrical cavities. Figure 3 shows the results of reconstruction of ellipsoidal and conical cavities ($\varepsilon_v = (|v_{\rm in} - v_{\rm rec}|/v_{\rm in}) \cdot 100\%$ is the relative error with which the volume of the cavity is found).

By solving Eq. (2.10), one can reconstruct the location and relative volume of an arbitrary cavity, knowing the displacement of the free end of the rod only at two frequencies. The results of the computational experiments are given in Table 1 ($\varepsilon_\alpha = (|\alpha_{\rm in} - \alpha_{\rm rec}|/\alpha_{\rm in}) \cdot 100$ % is the relative error with which the coordinate of the center of the cavity is reconstructed).

The data presented in the paper show that the approach proposed here is an effective one.

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